

Maximal amenability of the generator subalgebra in q -Gaussian von Neumann algebras

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Abstract

In this article, we give explicit examples of maximal amenable subalgebras of the q -Gaussian algebras, namely, the generator masa is maximal amenable inside the q -Gaussian algebras for real numbers q with its absolute value sufficiently small. To achieve this, we construct a Riesz basis in the spirit of Rădulescu [23] and develop a structural theorem for the q -Gaussian algebras.

Introduction

Starting with a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, Voiculescu's free probability theory associates each vector $\xi \in \mathcal{H}_{\mathbb{R}}$ with a semi-circular element $s(\xi)$, which is the sum of the left creation operator and the annihilation operator of ξ on the full Fock space of $\mathcal{H}_{\mathbb{R}}$. The von Neumann algebra generated by $\{s(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}$ turns out to be isomorphic to the free group factor $L(\mathbb{F}_{\dim(\mathcal{H}_{\mathbb{R}})})$, which is one of the central objects of the study of II_1 factors (see [27] for details). After Voiculescu's free Gaussian functor, three generalizations were developed: the first is Bożejko and Speicher's q -Gaussian functor for $-1 < q < 1$ in [5], the second is Shlyakhtenko's approach via quasi-free states in [25], and the third is Hiai's construction of q -deformed Araki-Woods algebras in [14] which is the combination of the first two. These constructions provide von Neumann algebras which can be seen as deformed free group factors. Hence it is natural to ask whether they have similar properties to those of the free group factors and whether they are isomorphic to the free group factors. These investigations will lead us to more profound understanding of the free group factors; they will help us understand what presentations they admit and what properties distinguish the free group factors from other factors.

In this paper, we will focus on the study of the structure of Bożejko and Speicher's q -Gaussian von Neumann algebras. These von Neumann algebras are currently under intense study by various authors, and many properties are known about them: assuming $\dim \mathcal{H}_{\mathbb{R}} \geq 2$, the q -Gaussian algebras are II_1 factors [24], non-injective [6, 18], strongly solid [1]. Moreover, when $\dim \mathcal{H}_{\mathbb{R}} < \infty$ and when $|q|$ is small (which depends on the dimension) the q -Gaussian algebras are isomorphic to free group factors [11, 13].

One notable aspect of a free group factor is that it contains a distinguished subalgebra, that is, the von Neumann subalgebra generated by one of the generators of the free group. It has many interesting properties: singularity, mixing property and maximal amenability. Since q -Gaussians are expected to have similar properties to those of the free group factors, it is natural to try to find a q -Gaussian counterpart of this subalgebra. In this context, we consider the generator subalgebra in a q -Gaussian algebra which is defined as the von Neumann subalgebra generated by $s(\xi)$, where $\xi \in \mathcal{H}_{\mathbb{R}}$ is an arbitrary fixed non-zero vector. This subalgebra coincides with the original generator subalgebra in free group factors as in [22] when $q = 0$. We also note that this subalgebra is already known to be quite useful: the factoriality of the q -Gaussian algebras when $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$ relies essentially on the fact that the generator subalgebra is a maximal abelian subalgebra (masa) as in [24]. Furthermore, recently it was shown that the generator subalgebra is singular [2, 29].

Another motivation to study the generator subalgebra comes from Peterson's conjecture about the structure of the free group factors. Recall that in [20] Ozawa and Popa showed the ground-breaking result that any free group factor $L(\mathbb{F}_n)$ is strongly solid, that is, if $B \subset L(\mathbb{F}_n)$ is a diffuse amenable subalgebra, then the normalizer of B inside $L(\mathbb{F}_n)$ again generates an amenable subalgebra. Strong solidity remains the deepest understanding of free group factors so far, implying other deep indecomposability results about the free group factors, such as lack of Cartan subalgebras due to Voiculescu [26], primeness due to Ge [12] and solidity due to Ozawa [19]. One possible generalization of strong solidity is proposed by Peterson (see the end of [21]):

Conjecture. *Any diffuse amenable subalgebra of a free group factor $L(\mathbb{F}_n)$ has a unique maximal amenable extension inside $L(\mathbb{F}_n)$.*

Evidences of Peterson's conjecture can be found in [17],[28],[8]. In particular, any generator subalgebra in a free group factor is the *unique* maximal amenable extension of any of its diffuse subalgebras.

In this paper, we prove that the generator subalgebra is in fact maximal amenable inside q -Gaussian algebras, when the absolute value of q is sufficiently small. Moreover, the result does not depend on the dimension of the real Hilbert space that we start with. In fact, our result confirms Peterson's conjecture for any diffuse subalgebra of the generator subalgebra:

Theorem. *Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space. Let q be any real number with $|q|$ sufficiently small and let A be a generator subalgebra of the q -Gaussian von Neumann algebra M associated with $\mathcal{H}_{\mathbb{R}}$. Then A is the unique maximal amenable extension of any of its diffuse subalgebras.*

This result can be seen as an evidence for the generator masa being the proper q -Gaussian counterpart of the generator masa of the free group factors. It is also notable that when $\dim \mathcal{H}_{\mathbb{R}} = \infty$ and $|q| \neq 0$ is small, this gives the first examples of maximal amenable masas of the q -Gaussian algebras. As shown in [13], when $\dim \mathcal{H}_{\mathbb{R}} < \infty$ and $|q| \neq 0$ is small,

$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is isomorphic to $L(\mathbb{F}_{\dim \mathcal{H}_{\mathbb{R}}})$. Even in this case, our main theorem is meaningful, namely, this possibly gives a family of examples supporting Peterson's conjecture which differs from the ones given in [17],[28],[8].

The proof relies on the notion of the *asymptotic orthogonality property* which is first introduced by Popa, [22] and used by many authors with great success (see for instance [7],[10],[3],[15],[16],[28],[8] and the references therein). In this strategy, the central theme is to show this property of the subalgebra. Basically, in order to show this property, one expands a vector of the standard Hilbert space along an appropriate basis and carries out some analyses. Historically, in order to do this, Popa [22] used the canonical basis coming from group elements and Houdayer [15] used the basis consisting of the basic tensors of the Fock space.

One may think that our situation is similar to the free-group-factor case and one may be tempted to simply mimic the proof for that case. However, a direct imitation does not resolve the problem: if we use the usual basis consisting of basic tensors, it is quite difficult to show this property however small the absolute value of q is. The problem is that as elements of the q -Fock space get into higher tensor parts, it becomes very hard to control the norms. As a result, the situation is completely different from that of the free-group-factor case and it is hopeless to carry out any analysis.

To avoid this difficulty, we have to construct a suitable basis. Our key idea comes from Rădulescu's work in [23]. There is a reason why we look at his construction; in [10], Cameron-Fang-Ravichandran-White showed the maximal amenability of a subalgebra which is called the *radial masa* of the free group factors using Radulescu's basis (Later, Wen simplified the proof in [28]). Although our situation is seemingly different from their situation, when the absolute value of q is small enough, it is possible to construct a basis suitable for our purpose.

By using this basis, we would like to develop a similar strategy to that of Wen [28]. Unfortunately, even after the basis is constructed, the subsequent computations are arduous; some key facts when $q = 0$ are no longer true when q is non-zero. Nevertheless, we expect that what are true in the case of $q = 0$ should be "approximatively true" for the general q -Fock spaces, at least when $|q|$ is small. In this spirit, we are able to carry out the analyses.

The paper consists of 5 sections and an appendix. Section 1 is about preliminaries for the q -Fock space and there we establish some notations which will be used throughout the paper. In Section 2 we develop some q -combinatorics for later use. The next two sections will take up the majority of our paper: Section 3 introduces and proves the Rădulescu type of Rietz basis and Section 4 contains the key estimates for elements in the relative commutant of A in the ultraproduct. In the last section, we establish the strong asymptotic orthogonality property for the inclusion of the generator subalgebra $A \subset M$ and finishes the proof of the main theorem. One can also recover the fullness of those q -Gaussian algebras using the basis we constructed. Finally, in the appendix, we include an observation that the strong asymptotic orthogonality property implies singularity for diffuse abelian subalgebras.

Since we will use the singularity of the generator masa in the proof of the strong asymptotic orthogonality property, this result does not make our argument easier. However, this result will save us the trouble of showing the singularity once we know the strong asymptotic orthogonality property. Hence we include it in the end of the paper.

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1 Preliminaries

Throughout the paper we assume that $-1 < q < 1$ and let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with $\dim \mathcal{H}_{\mathbb{R}} \geq 2$. Denote by $\mathcal{H} := \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathcal{H}_{\mathbb{R}}$. Define an inner product on $\bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$ by

$$\langle e_1 \otimes \cdots \otimes e_n, f_1 \otimes \cdots \otimes f_m \rangle_q = \delta_n(m) \sum_{\sigma \in S_m} q^{|\sigma|} \langle e_1 \otimes \cdots \otimes e_n, f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)} \rangle,$$

where S_m is the group of permutations on $\{1, \dots, m\}$, $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ is the space spanned by the vacuum vector Ω , the inner product on the right-hand side is the usual one on $\mathcal{H}^{\otimes m}$ and by $|\sigma|$ we mean the number of inversions of $\sigma \in S_m$ given by

$$|\sigma| = \#\{(i, j) \in \{1, \dots, m\}^2 : i < j, \sigma(i) > \sigma(j)\}.$$

The q -deformed Fock space $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ is the completion of $(\bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}, \langle \cdot, \cdot \rangle_q)$ and $\|\cdot\|_q$ is the norm induced by this inner product. For simplicity, sometimes we will suppress the subscript q for the inner product and the norm on the q -Fock space.

For $e \in \mathcal{H}_{\mathbb{R}}$, we define the *left creation operator* $c(e)$ and the *right creation operator* $c_r(e)$ on $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ by $c(e)(\Omega) = e = c_r(e)(\Omega)$ and

$$\begin{aligned} c(e)(e_1 \otimes \cdots \otimes e_n) &= e \otimes e_1 \otimes \cdots \otimes e_n, \\ c_r(e)(e_1 \otimes \cdots \otimes e_n) &= e_1 \otimes \cdots \otimes e_n \otimes e, \end{aligned} \tag{1}$$

for $n \geq 1$. Both $c(e)$ and $c_r(e)$ are bounded operators [5, Lemma 4] and their adjoints $a(e) = c^*(e)$, $a_r(e) = c_r^*(e)$ are called the *left annihilation operator* and the *right annihilation operator*.

operator, respectively, which are given by $a(e)(\Omega) = 0 = a_r(e)(\Omega)$ and

$$\begin{aligned} a(e)(e_1 \otimes \cdots \otimes e_n) &= \sum_{1 \leq i \leq n} q^{(i-1)} \langle e, e_i \rangle e_1 \otimes \cdots \otimes \hat{e}_i \otimes \cdots \otimes e_n, \\ a_r(e)(e_1 \otimes \cdots \otimes e_n) &= \sum_{1 \leq i \leq n} q^{(n-i)} \langle e, e_i \rangle e_1 \otimes \cdots \otimes \hat{e}_i \otimes \cdots \otimes e_n, \end{aligned} \quad (2)$$

for $n \geq 1$, where \hat{e}_i means a removed letter. Note that $c(e)$ and $c_r(f)$ commute but $c(e)$ and $a_r(f)$ do not commute in general.

The operators $c(e), c_r(e)$ satisfy the following important *q-commutatiton relations* [5]:

$$\begin{aligned} a(e)c(f) - qc(f)a(e) &= \langle e, f \rangle Id, \\ a_r(e)c_r(f) - qc_r(f)a_r(e) &= \langle e, f \rangle Id. \end{aligned} \quad (3)$$

For $e \in \mathcal{H}_{\mathbb{R}}$, let

$$s(e) = c(e) + a(e)$$

and let $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ be the von Neumann algebra generated by $\{s(e) : e \in \mathcal{H}_{\mathbb{R}}\}$. We call it the *q-Gaussian algebra associated with $\mathcal{H}_{\mathbb{R}}$* . It is known that $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is a type II_1 factor [24, Corollary 1] and Ω is a generating and cyclic vector which gives the trace for $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ [6, Theorem 4.3, 4.4]. Consequently, each element $x \in \Gamma_q(\mathcal{H}_{\mathbb{R}})$ is uniquely determined by $\xi = x \cdot \Omega \in \mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ and we write $x = s(\xi)$. This notation is consistent with the above definition for $s(e), e \in \mathcal{H}_{\mathbb{R}}$.

One can also define $s_r(e) = c_r(e) + a_r(e)$ for $e \in \mathcal{H}_{\mathbb{R}}$ and define $\Gamma_{q,r}(\mathcal{H}_{\mathbb{R}}) := \{s_r(e) : e \in \mathcal{H}_{\mathbb{R}}\}''$. Then similar to the group von Neumann algebra case we have $\Gamma_{q,r}(\mathcal{H}_{\mathbb{R}}) = \Gamma_q(\mathcal{H}_{\mathbb{R}})'$.

Here we record two facts that will be used in this paper.

- Let $e \in \mathcal{H}$ be a unit vector, then

$$\|e^{\otimes n}\|_q^2 = [n]_q!, \quad (4)$$

where $[k]_q = \frac{1-q^k}{1-q}$ and $[n]_q! = [1]_q \cdots [n]_q$. We also define $[0]_q! := 1$.

- (**Wick formula**, [4, Proposition 2.7]) Let $e_1 \otimes \cdots \otimes e_n \in \mathcal{H}^{\otimes n}$, then

$$\begin{aligned} s(e_1 \otimes \cdots \otimes e_n) &= \sum_{i=0}^n \sum_{\sigma \in S_n / (S_{n-i} \times S_i)} q^{|\sigma|} c(e_{\sigma(1)}) \cdots c(e_{\sigma(n-i)}) \\ &\quad \times a(e_{\sigma(n-i+1)}) \cdots a(e_{\sigma(n)}), \end{aligned} \quad (5)$$

where σ is the representative of $S_{n-i} \times S_i$ in S_n with minimal number of inversions.

From now on we fix a unit vector $e \in \mathcal{H}_{\mathbb{R}}$ and we call the von Neumann subalgebra $\Gamma_q(\mathbb{R}e) \subset \Gamma_q(\mathcal{H}_{\mathbb{R}})$ a *generator subalgebra*. It is shown by Ricard in [24] that this gives a maximal abelian subalgebra (masa) of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$.

Let $T : \mathcal{H}_{\mathbb{R}} \rightarrow \mathcal{H}_{\mathbb{R}}$ be a \mathbb{R} -linear contraction. We still denote by T its complexification given by $T(\xi + i\eta) = T(\xi) + iT(\eta)$, $\forall \xi, \eta \in \mathcal{H}_{\mathbb{R}}$. Then the *first quantization* $\mathcal{F}_q(T)$, is the bounded operator on $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ defined by

$$\mathcal{F}_q(T) = Id_{\mathbb{C}\Omega} \oplus \bigoplus_{n \geq 1} T^{\otimes n}.$$

The *second quantization* of T , is the unique unital completely positive map $\Gamma_q(T)$ on $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ defined as

$$\Gamma_q(T)(s(\xi)) = s(\mathcal{F}_q(T)(\xi)).$$

In particular, if $T = E_e : \mathcal{H}_{\mathbb{R}} \rightarrow \mathbb{R}e$ is the orthogonal projection, then $\mathcal{F}_q(E_e)$ is the conditional expectation of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ onto $\Gamma_q(\mathbb{R}e)$.

To simplify notations, from now on we will write $A := \Gamma_q(\mathbb{R}e)$ for the generator subalgebra and $M := \Gamma_q(\mathcal{H}_{\mathbb{R}})$ for the q -Gaussian algebra.

2 Some q -combinatorics

In this section we will develop some formulas about combinatorics in q -Gaussians that will be needed in later sections.

For $n, m \in \mathbb{N} \cup \{0\}$, $n \geq m$, set

$$\binom{n}{m}_q = \frac{[n]_q!}{[m]_q! \cdot [n-m]_q!} = \prod_{i=1}^{n-m} \frac{1 - q^{m+i}}{1 - q^i}.$$

We make the following convention

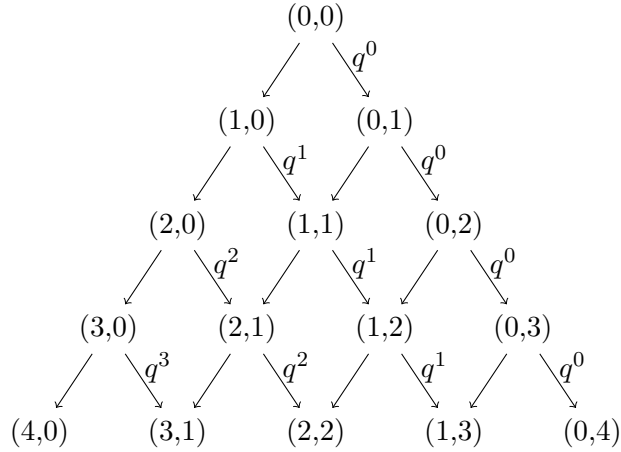
$$\binom{n}{m}_q = 0 \quad \text{whenever } m > n \text{ or } m < 0.$$

The following q -analogue of the Pascal's identity for these q -binomial coefficients (c.f. [4, Proposition 1.8]) can be easily checked:

Lemma 1. *For all $m \in \mathbb{Z}$ and $n \geq 0$,*

$$\binom{n+1}{m}_q = q^m \binom{n}{m}_q + \binom{n}{m-1}_q = \binom{n}{m}_q + q^{n-m+1} \binom{n}{m-1}_q. \quad (6)$$

Continuing the analogy, the q -binomial coefficients $\binom{n}{m}_q$ can also be seen to count ‘number’ of weighted paths in the ‘ q -Pascal’s triangle’ from $(0,0)$ to $(n-m, m)$. The q -Pascal triangle is formed from the ordinary Pascal triangle by putting a weight of q^i on each (right) edge from (i, j) to $(i, j+1)$, as shown below. All the other (left) edges will have weight 1. The weight of a path is the product of the weights on its constituent edges.



For instance, the sum of all weighted paths from $(0,0)$ to $(1,2)$ is $1+q+q \cdot q = [3]_q = \binom{3}{2}_q$. It is clear from the diagram that they satisfy the second recurrence relation mentioned above, with the other one following from the symmetry of the q -binomial coefficient.

Lemma 2. For $n_1, n_2, m \in \mathbb{N} \cup \{0\}$ with $n_1 + n_2 \geq m$, we have

$$\sum_{i=0}^m q^{(n_1-i)(m-i)} \binom{n_1}{i}_q \binom{n_2}{m-i}_q = \binom{n_1+n_2}{m}_q. \quad (7)$$

Proof. Any path from $(0,0)$ to (n_1+n_2-m, m) will pass through (n_1-i, i) for some $0 \vee m-n_2 \leq i \leq m \wedge n_1$. This range of index corresponds exactly to the terms with non-zero contribution in the above sum. Now $\binom{n_1}{i}_q$ counts the sum of weighted paths from $(0,0)$ to (n_1-i, i) . To go from (n_1-i, i) to (n_1+n_2-m, m) involves travelling along paths counted by $\binom{n_2}{m-i}_q$.

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Now suppose that (9) holds up to m, n . Then

$$\begin{aligned}
X^{m+1}Y^n &= \sum_{i=0}^m q^{(n-i)(m-i)} [i]_q! \binom{n}{i}_q \binom{m}{i}_q XY^{n-i} X^{m-i} \\
&= \sum_{i=0}^m \left(q^{(n-i)(m-i+1)} [i]_q! \binom{n}{i}_q \binom{m}{i}_q Y^{n-i} X^{m+1-i} \right. \\
&\quad \left. + q^{(n-i)(m-i)} [i]_q! \binom{n}{i}_q \binom{m}{i}_q [n-i]_q Y^{n-1-i} X^{m-i} \right) \\
&= \sum_{i=0}^{m+1} Y^{n-i} X^{m+1-i} \left(q^{(n-i)(m+1-i)} [i]_q! \binom{n}{i}_q \binom{m}{i}_q \right. \\
&\quad \left. + q^{(n-i+1)(m+1-i)} [i-1]_q! \binom{n}{i-1}_q \binom{m}{i-1}_q [n+1-i]_q \right) \\
&= \sum_{i=0}^{m+1} Y^{n-i} X^{m+1-i} q^{(n-i)(m+1-i)} [i]_q! \binom{n}{i}_q \left(\binom{m}{i}_q + q^{m+1-i} \binom{m}{i-1}_q \right) \\
&= \sum_{i=0}^{m+1} Y^{n-i} X^{m+1-i} q^{(n-i)(m+1-i)} [i]_q! \binom{n}{i}_q \binom{m+1}{i}_q.
\end{aligned}$$

Here in the second equation we used (8), the third equality is due to the simple fact

$$[i-1]_q! \binom{n}{i-1}_q [n-i+1]_q = [i]_q! \binom{n}{i}_q,$$

and the last equality comes from (6).

The case for $X^m Y^{n+1}$ is completely similar so we omit the details. □

Now we turn to the relation between powers of X and Z . Naturally, W also comes into play.

Lemma 5. *For $m, n \in \mathbb{N}$, we have*

$$X^m Z^n = \sum_{i=0}^m [i]_q! \binom{n}{i}_q \binom{m}{i}_q Z^{n-i} W^i X^{m-i}. \quad (10)$$

Proof. Let's proceed by induction on m and n .

One can easily show that

$$\begin{aligned}
XZ^n &= Z^n X + [n]_q Z^{n-1} W, \\
X^m Z &= Z X^m + [m]_q W X^{m-1},
\end{aligned}$$

which are special cases for (10).

Suppose that (10) holds up to m, n . Then

$$\begin{aligned}
X^{m+1}Z^n &= \sum_{i=0}^m [i]_q! \binom{n}{i}_q \binom{m}{i}_q XZ^{n-i}W^iX^{m-i} \\
&= \sum_{i=0}^m [i]_q! \binom{n}{i}_q \binom{m}{i}_q (Z^{n-i}X + [n-i]_q Z^{n-i-1}W) W^i X^{m-i} \\
&= \sum_{i=0}^m [i]_q! \binom{n}{i}_q \binom{m}{i}_q (q^i Z^{n-i}W^i X^{m+1-i} + [n-i]_q Z^{n-i-1}W^{i+1} X^{m-i}) \\
&= \sum_{i=0}^{m+1} Z^{n-i}W^i X^{m+1-i} \left([i]_q! \binom{n}{i}_q \binom{m}{i}_q q^i + [i-1]_q! \binom{n}{i-1}_q \binom{m}{i-1}_q [n-i+1]_q \right) \\
&= \sum_{i=0}^{m+1} [i]_q! \binom{n}{i}_q \binom{m+1}{i}_q Z^{n-i}W^i X^{m+1-i}.
\end{aligned}$$

The other case is completely similar. \square

Proposition 6. *Let $X = a(e), Y = c(e), Z = c_r(e)$ and W be the bounded operator on $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ defined by*

$$W(\xi) = q^n \xi, \forall \xi \in \mathcal{H}^{\otimes n}.$$

Then X, Y, Z, W satisfy the relations listed in (8). Consequently, (9) and (10) hold true.

Proof. (8) can be checked by direct computations hence (9) and (10) follow from the previous two lemmas. \square

3 Rădulescu basis in q -Fock spaces

In this section we construct the Rădulescu basis for $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$, which will be the fundamental tool to study the generator subalgebra. The construction is motivated by the original construction of Rădulescu in [23].

For each integer $k \geq 0$, we consider the following subspace of $\mathcal{H}^{\otimes k} \subset \mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$:

$$T^k := \{\xi \in \mathcal{H}^{\otimes k} : a(e)\xi = a_r(e)\xi = 0\}. \quad (11)$$

It is clear that $T^0 = \mathbb{C}\Omega$ and each T^k is non-zero (for instance, if we choose $f \in \mathcal{H}_{\mathbb{R}}$ with $f \perp e$, then $f^{\otimes k} \in T^k$).

For each $\xi \in \mathcal{H}^{\otimes k}$ and for all $s, t \in \mathbb{N} \cup \{0\}$, define

$$\xi_{s,t} = e^{\otimes s} \otimes \xi \otimes e^{\otimes t} \in \mathcal{H}^{\otimes (k+s+t)}. \quad (12)$$

We also make the convention that $\xi_{s,t} = 0$ if either $s < 0$ or $t < 0$.

We start with a few important observations.

Lemma 7. *For $\xi \in T^k$ and $s, t \geq 0$, we have the following*

$$\begin{aligned} a(e)\xi_{s,t} &= [s]_q \xi_{s-1,t} + q^{s+k} [t]_q \xi_{s,t-1}, \\ a_r(e)\xi_{s,t} &= q^{t+k} [s]_q \xi_{s-1,t} + [t]_q \xi_{s,t-1}. \end{aligned} \quad (13)$$

Proof. Since the left and right annihilation operators behave similarly, we just show the first equation. This is a consequence of the following equation

$$\begin{aligned} a(e)(W_1 \otimes W_2 \otimes W_3) &= (a(e)W_1) \otimes W_2 \otimes W_3 + q^{|W_1|} W_1 \otimes (a(e)W_2) \otimes W_3 \\ &\quad + q^{|W_1|+|W_2|} W_1 \otimes W_2 \otimes (a(e)W_3), \end{aligned}$$

where $W_i, i = 1, 2, 3$ are basic words and $|W_i|$ stands for the length. By linearity, the equation still holds even if W_2 is a linear combination of basic words with the same length. Thus for $\xi \in T^k$, we have

$$\begin{aligned} a(e)\xi_{s,t} &= a(e)(e^{\otimes s} \otimes \xi \otimes e^{\otimes t}) \\ &= (a(e)e^{\otimes s}) \otimes \xi \otimes e^{\otimes t} + q^s e^{\otimes s} \otimes (a(e)\xi) \otimes e^{\otimes t} + q^{s+k} e^{\otimes s} \otimes \xi \otimes (a(e)e^{\otimes t}) \\ &= [s]_q \xi_{s-1,t} + 0 + q^{s+k} [t]_q \xi_{s,t-1}. \end{aligned}$$

□

Lemma 8. *For $\xi \in T^k$, we have*

$$\begin{aligned} s(e)^n s_r(e)^m \xi &\in \text{span}\{\xi_{s,t} : s, t \geq 0\}, \forall n, m \geq 0, \\ \xi_{s,t} &\in \text{span}\{s(e)^n s_r(e)^m \xi : n, m \geq 0\}, \forall s, t \geq 0. \end{aligned} \quad (14)$$

Proof. For the first inclusion,

$$\begin{aligned} s(e)^n s_r(e)^m \xi &= (c(e) + a(e))^n (c_r(e) + a_r(e))^m \xi \\ &= (c_r(e) + a_r(e))^m (c(e) + a(e))^n \xi. \end{aligned} \quad (15)$$

Application of the q -commutation relations implies that we can write $(c(e) + a(e))^n$ and $(c_r(e) + a_r(e))^m$ as polynomials of the form

$$(c(e) + a(e))^n = \sum_{i,j \geq 0} a_{i,j} c(e)^i a(e)^j, \quad (c_r(e) + a_r(e))^m = \sum_{k,l \geq 0} b_{k,l} c_r(e)^k a_r(e)^l. \quad (16)$$

Thus $s(e)^n s_r(e)^m \xi$ is a linear combination of

$$c(e)^i a(e)^j c_r(e)^k a_r(e)^l \xi,$$

where $i, j, k, l \in \mathbb{N} \cup \{0\}$. By the previous lemma, all such terms are in $\text{span}\{\xi_{s,t} : s, t \geq 0\}$, which yields the first inclusion.

We prove the second inclusion by inducting on $s + t$. When $s + t = 0$, the conclusion clearly holds. Suppose now that the inclusion holds for $s + t \leq N$. By Lemma 7 we have that

$$\begin{aligned} s(e)\xi_{s,t} &= \xi_{s+1,t} + [s]_q \xi_{s-1,t} + q^{s+k} [t]_q \xi_{s,t-1}, \\ s_r(e)\xi_{s,t} &= \xi_{s,t+1} + q^{t+k} [s]_q \xi_{s-1,t} + [t]_q \xi_{s,t-1}. \end{aligned}$$

Hence the conclusion holds for $s + t = N + 1$ as well. \square

For $k \geq 0$, let

$$Q_k : \mathcal{F}_q(\mathcal{H}_{\mathbb{R}}) \rightarrow \mathcal{H}^{\otimes k},$$

be the orthogonal projections from the q -Fock space onto $\mathcal{H}^{\otimes k}$ and we define

$$S^k := \mathcal{H}^{\otimes k} \ominus T^k.$$

For notational convenience, we let $\mathcal{H}^{\otimes i} = \{0\}$, for all $i < 0$.

We first characterize S^k as follows:

Lemma 9. *For all $k \geq 0$, $S^k = \text{span}\{Q_k(s(e)\eta), Q_k(s_r(e)\eta) : \eta \in \mathcal{H}^{\otimes l}, l < k\}$.*

Proof. It suffices to show that $\xi \in \mathcal{H}^{\otimes k}$ belongs to T^k if and only if

$$\langle \xi, Q_k(s(e)\eta) \rangle_q = \langle \xi, Q_k(s_r(e)\eta) \rangle_q = 0$$

for any $\eta \in \mathcal{H}^{\otimes l}, l < k$.

To see this, notice that

$$\langle \xi, Q_k(s(e)\eta) \rangle_q = \langle \xi, Q_k(c(e)\eta) \rangle_q = \langle a(e)\xi, \eta \rangle_q.$$

Since $a(e)\xi \in \mathcal{H}^{\otimes(k-1)}$, we have that $\langle \xi, Q_k(s(e)\eta) \rangle_q = 0$ for any $\eta \in \mathcal{H}^{\otimes l}, l < k$ if and only if $a(e)\xi = 0$.

Similarly, $\langle \xi, Q_k(s_r(e)\eta) \rangle_q = 0$ for any $\eta \in \mathcal{H}^{\otimes l}, l < k$ if and only if $a_r(e)\xi = 0$. \square

Lemma 10. *For all $k \geq 0$, $\mathcal{H}^{\otimes k} \subset \text{span}\{s(e)^n s_r(e)^m \xi : \xi \in T^l, l \leq k, n, m \geq 0\}$.*

Proof. We prove it by induction. When $k = 0$, the statement is clearly true. Assume that the lemma holds up to $k - 1$ and let $\eta \in \mathcal{H}^{\otimes k}$. We may further assume that $\eta \in S^k$.

By Lemma 9, η is a linear combination of $Q_k(s(e)\xi)$ and $Q_k(s_r(e)\xi)$, $\xi \in \mathcal{H}^{\otimes(k-1)}$. By the induction hypothesis, each $\xi \in \mathcal{H}^{\otimes(k-1)}$ is a linear combination of $s(e)^n s_r(e)^m \xi'$, $\xi' \in T^l$, $l \leq k - 1$, $n, m \geq 0$. Thus η is a linear combination of $Q_k(s(e)^n s_r(e)^m \xi')$, $\xi' \in T^l$, $l \leq k - 1$, $n, m \geq 0$.

Now, by Lemma 8, $Q_k(s(e)^n s_r(e)^m \xi') \in \text{span}\{\xi'_{r,s} : r, s \geq 0, r + s + |\xi'| = k\}$ but again by Lemma 8, each $\xi'_{r,s} \in \text{span}\{s(e)^n s_r(e)^m \xi' : n, m \geq 0\}$. Therefore, we are done. \square

Lemma 11. *Suppose that $\xi \in T^t$ and $r, s, k \geq 0$ are non-negative integers, then we have*

$$\begin{aligned} a(e)^k \xi_{r,s} &= \sum_{i+j=k, i,j \geq 0} \frac{[r]_q!}{[r-j]_q!} \cdot \frac{[s]_q!}{[s-i]_q!} \cdot \binom{k}{i}_q q^{(t+r-j)i} \xi_{r-j, s-i}, \\ a_r(e)^k \xi_{r,s} &= \sum_{i+j=k, i,j \geq 0} \frac{[r]_q!}{[r-i]_q!} \cdot \frac{[s]_q!}{[s-j]_q!} \cdot \binom{k}{i}_q q^{(t+s-j)i} \xi_{r-i, s-j}. \end{aligned} \quad (17)$$

Proof. This is just an induction via direct computations so we omit the details. \square

Now we compute the inner products between $\xi_{r,s}$.

Lemma 12. *Let $\xi, \eta \in \cup_{k \geq 0} T^k$ with $\xi \perp \eta$, then for any r, s, r', s' non-negative integers, we have*

$$\langle \xi_{r,s}, \eta_{r',s'} \rangle = 0. \quad (18)$$

Moreover, if $r + s \neq r' + s'$, then

$$\langle \xi_{r,s}, \xi_{r',s'} \rangle = 0.$$

Proof. The second statement is trivial so we focus on the first. By Lemma 8 and the fact that $s(e)s_r(e) = s_r(e)s(e)$, it suffices to show that

$$s(e)^n s_r(e)^m \xi \perp \eta$$

for all n, m non-negative integers. Again by Lemma 8, it reduces to show that

$$\xi_{r,s} \perp \eta$$

for all r, s non-negative integers. This is clear by the definition of T^k unless $r = s = 0$, but then the assumption $\xi \perp \eta$ leads to the conclusion. \square

Proposition 13. *Let r, s, r', s', k be non-negative integers with $r + s = r' + s'$ and $\xi \in T^k$ of norm 1, then*

$$\langle \xi_{r,s}, \xi_{r',s'} \rangle = \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q. \quad (19)$$

Proof. The proof is simply a direct but lengthy computation. However for the convenience of the readers, we include the details here. For simplicity, we write $\alpha_{n,m}^i := [i]_q! \binom{n}{i}_q \binom{m}{i}_q$ for $n, m, i \geq 0$. Note that by our convention, $\alpha_{n,m}^i = 0$ when either $i > n$ or $i > m$.

Now we compute

$$\begin{aligned}
\langle \xi_{r,s}, \xi_{r',s'} \rangle &= \langle c(e)^r c_r(e)^s \xi, c(e)^{r'} c_r(e)^{s'} \xi \rangle \\
&= \langle a(e)^{r'} c(e)^r c_r(e)^s \xi, c_r(e)^{s'} \xi \rangle \\
&\stackrel{(9)}{=} \left\langle \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \alpha_{r,r'}^i c(e)^{r-i} a(e)^{r'-i} c_r(e)^s \xi, c_r(e)^{s'} \xi \right\rangle \\
&\stackrel{(10)}{=} \left\langle \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \alpha_{r,r'}^i c(e)^{r-i} \sum_{j=0}^{r'-i} \alpha_{r'-i,s}^j c_r(e)^{s-j} W^j a(e)^{r'-i-j} \xi, c_r(e)^{s'} \xi \right\rangle \\
&\stackrel{(*)}{=} \left\langle \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \alpha_{r,r'}^i c(e)^{r-i} \alpha_{r'-i,s}^{r'-i} c_r(e)^{s-(r'-i)} W^{r'-i} \xi, c_r(e)^{s'} \xi \right\rangle \\
&= \sum_{i=0}^{r'} q^{(r-i)(r'-i)} q^{(r'-i)k} \alpha_{r,r'}^i \alpha_{r'-i,s}^{r'-i} \left\langle c_r(e)^{s-(r'-i)} \xi, a(e)^{r-i} c_r(e)^{s'} \xi \right\rangle \\
&\stackrel{(10)}{=} \sum_{i=0}^{r'} q^{(r-i)(r'-i)} q^{(r'-i)k} \alpha_{r,r'}^i \alpha_{r'-i,s}^{r'-i} \left\langle c_r(e)^{s-(r'-i)} \xi, \sum_{j=0}^{r-i} \alpha_{r-i,s'}^j c_r(e)^{s'-j} W^j a(e)^{r-i-j} \xi \right\rangle \\
&\stackrel{(*)}{=} \sum_{i=0}^{r'} q^{(r-i)(r'-i)} q^{(r'-i)k} q^{(r-i)k} \alpha_{r,r'}^i \alpha_{r'-i,s}^{r'-i} \alpha_{r-i,s'}^{r-i} \left\langle c_r(e)^{s-(r'-i)} \xi, c_r(e)^{s'-(r-i)} \xi \right\rangle \\
&= \sum_{i=0}^{r'} q^{(r-i)(r'-i)} q^{(r'-i)k} q^{(r-i)k} \alpha_{r,r'}^i \alpha_{r'-i,s}^{r'-i} \alpha_{r-i,s'}^{r-i} [s - r' + i]_q! \\
&= \sum_{i=0}^{r'} q^{(r-i)(r'-i) + k(r-i) + k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q,
\end{aligned}$$

where in the equations with $(*)$ we used the fact that $a(e)\xi = 0$.

□

For later use, we define two constants depending on q :

$$C(q) := \prod_{i=1}^{\infty} \frac{1}{1 - q^i}, \quad D(q) := \prod_{i=1}^{\infty} (1 + |q|^i).$$

Basic calculus shows that whenever $-1 < q < 1$, the above two limits exist unconditionally.

We record a simple but very useful estimate here for later references.

Lemma 14. *For all $-1 < q < 1$ and for all $n, m \geq 0$, we have*

$$\left| \binom{n}{m}_q \right|^{\pm 1} \leq D(q)C(|q|).$$

Lemma 15. *Let r, s, r', s', k be non-negative integers with $r + s = r' + s', r \geq r'$ and $\xi \in T^k$ of norm 1. Then for each $-1 < q < 1$, there are constants $E(q), F(q)$ such that*

$$E(q)|q|^{k(r-r')}[r+s]_q! \leq |\langle \xi_{r,s}, \xi_{r',s'} \rangle| \leq F(q)|q|^{k(r-r')}[r+s]_q!. \quad (20)$$

Moreover, we have that

$$\lim_{q \rightarrow 0} E(q) = \lim_{q \rightarrow 0} F(q) = 1.$$

Proof. When $0 \leq q < 1$, we have

$$\begin{aligned} \langle \xi_{r,s}, \xi_{r',s'} \rangle &= \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\ &\leq q^{k(r-r')} \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\ &\stackrel{(7)}{=} q^{k(r-r')} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r+s}{r'}_q \\ &= q^{k(r-r')} [r+s]_q!, \end{aligned}$$

where the last equality comes from the assumption $r + s = r' + s'$. This proves the inequality on the right side.

For the inequality on the left side, simply note that

$$\begin{aligned} &\sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\ &\geq \sum_{i=r'}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\ &= q^{k(r-r')} [r']_q! [s']_q! \cdot \binom{r}{r'}_q \\ &= q^{k(r-r')} [r']_q! [s']_q! \cdot \frac{[r]_q!}{[r']_q! [r-r']_q!} \cdot \frac{[r+s]_q!}{[r+s]_q!} \end{aligned}$$

$$\begin{aligned}
&= q^{k(r-r')} [r+s]_q! \cdot \frac{[s']_q! [r]_q!}{[r-r']_q! [r+s]_q!} \\
&= q^{k(r-r')} [r+s]_q! \cdot \frac{(1-q^{r-r'+1}) \cdots (1-q^r)}{(1-q^{s'+1}) \cdots (1-q^{r+s})} \\
&\geq q^{k(r-r')} [r+s]_q! \cdot (1-q^{r-r'+1}) \cdots (1-q^r) \\
&\geq \frac{1}{C(q)} q^{k(r-r')} [r+s]_q!.
\end{aligned}$$

Thus if we let $E(q) = \frac{1}{C(q)}$, $F(q) = 1$, we are done.

Now assume $-1 < q < 0$. By Lemma 14 we have

$$[r']_q! [s']_q! \leq [r'+s']_q! D(q) C(|q|).$$

Therefore,

$$\begin{aligned}
&\left| \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \right| \\
&\leq D(q)^3 C(|q|)^3 [r'+s']_q! \sum_{i=0}^{r'} |q|^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \\
&\leq D(q)^3 C(|q|)^3 [r'+s']_q! \cdot |q|^{k(r-r')} \cdot \frac{1}{1-|q|}.
\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}
&\left| \sum_{i=r'}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \right| \\
&= |q|^{k(r-r')} [r']_q! \cdot [s']_q! \cdot \binom{r}{r'}_q \\
&= |q|^{k(r-r')} [r+s]_q! \cdot \frac{(1-q^{r-r'+1}) \cdots (1-q^r)}{(1-q^{s'+1}) \cdots (1-q^{r+s})} \\
&\geq |q|^{k(r-r')} [r+s]_q! \cdot \frac{(1-|q|^{r-r'+1}) \cdots (1-|q|^r)}{(1+|q|^{s'+1}) \cdots (1+|q|^{r+s})} \\
&\geq \frac{|q|^{k(r-r')}}{D(q)C(|q|)} \cdot [r+s]_q!,
\end{aligned}$$

and

$$\left| \sum_{i=0}^{r'-1} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \right|$$

$$\begin{aligned}
&\leq \sum_{i=0}^{r'-1} |q|^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\
&\leq |q|^{k(r-r')} \sum_{i=0}^{r'-1} |q|^{(r-i)} [r+s]_q! D(q)^3 C(|q|)^3 \\
&\leq |q|^{k(r-r')} \frac{|q|}{1-|q|} D(q)^3 C(|q|)^3 [r+s]_q!.
\end{aligned}$$

Hence by the triangle inequality,

$$\begin{aligned}
&\left| \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \right| \\
&\geq \frac{|q|^{k(r-r')}}{D(q)C(|q|)} \cdot [r+s]_q! - \frac{|q|^{k(r-r')+1}}{1-|q|} D(q)^3 C(|q|)^3 [r+s]_q! \\
&= |q|^{k(r-r')} \left(\frac{1}{D(q)C(|q|)} - \frac{|q|}{1-|q|} D(q)^3 C(|q|)^3 \right) \cdot [r+s]_q!.
\end{aligned}$$

Finally, if we let $E(q) = \frac{1}{D(q)C(|q|)} - \frac{|q|}{1-|q|} D(q)^3 C(|q|)^3$, $F(q) = \frac{D(q)^3 C(|q|)^3}{1-|q|}$, the proof is complete. \square

The following corollary will be used multiple times later.

Corollary 16. *Let r, s, k be non-negative integers and let $\xi \in T^k$ be of norm 1. Then there is a positive number $\alpha > 0$, such that whenever $|q| \leq \alpha$, we have*

$$\frac{1}{2} [r+s]_q! \leq \|\xi_{r,s}\|_q^2 \leq 2 [r+s]_q!. \quad (21)$$

Remark 17. Lemma 15 and Corollary 16 hold when $-1/7 < q < 1/4$.

To prove the main theorem of this section, we need another lemma.

Lemma 18. *Let $\alpha \in \mathbb{R}$ with $|\alpha| < 1$. For any $n \in \mathbb{N}$ we define*

$$E_\alpha = \begin{pmatrix} 0 & -\alpha & \cdots & -\alpha^{n-1} \\ -\alpha & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\alpha \\ -\alpha^{n-1} & \cdots & -\alpha & 0 \end{pmatrix},$$

then the operator norm $\|E_\alpha\| \leq \frac{2|\alpha|}{1-|\alpha|}$.

Proof. Clearly we have

$$\left\| \begin{pmatrix} 0 & -\alpha^k & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -\alpha^k \\ 0 & & & 0 \end{pmatrix} \right\| = |\alpha|^k.$$

Hence $\|E_\alpha\| \leq 2 \sum_{i=1}^{n-1} |\alpha|^i \leq \frac{2|\alpha|}{1-|\alpha|}$. □

Take an orthonormal basis $\{\xi_j^i : j \in I_k\}$ for $T_k, k \geq 1$. We may re-order the set $\cup_{i \in \mathbb{N}} \{\xi_j^i : j \in I_k\}$ as $\{\xi^i : i \in I\}$ for some index set I and we set that $\xi^0 = \Omega$.

Finally we are ready to state and prove the main result of this section.

Theorem 19. *For $-1 < q < 1$ with $|q|$ sufficiently small, the set $\left\{ \frac{\xi_{r,s}^i}{\|\xi_{r,s}^i\|} : i \in I, r, s \geq 0 \right\}$ forms a Rietz basis for $L^2(M) \ominus L^2(A)$, i.e., $\text{span} \left\{ \frac{\xi_{r,s}^i}{\|\xi_{r,s}^i\|} : i \in I, r, s \geq 0 \right\}$ is dense in $L^2(M) \ominus L^2(A)$ and there exists some constants $A_q, B_q > 0$, such that for all $\lambda_{r,s}^i \in \mathbb{C}$, one has*

$$A_q \sum_{r,s,i} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2 \leq \left\| \sum_{r,s,i} \lambda_{r,s}^i \xi_{r,s}^i \right\|^2 \leq B_q \sum_{r,s,i} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2. \quad (22)$$

Proof. By Lemma 12, it suffices to find such $A_q, B_q > 0$ which are independent of $i \in I$ and $k \geq 0$ such that

$$A_q \sum_{r+s=k} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2 \leq \left\| \sum_{r+s=k} \lambda_{r,s}^i \xi_{r,s}^i \right\|^2 \leq B_q \sum_{r+s=k} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2$$

holds for any $\lambda_{r,s}^i \in \mathbb{C}$. Fixing $\xi = \xi^i \in T^t$ for some $t \in \mathbb{N}$, for simplicity, we will omit the superscript i in the rest of the proof. We fix an $\epsilon > 0$ small enough such that $|q| \leq \epsilon$ implies

that $1/2 \leq E(q), F(q) \leq 2$ (in particular, Corollary 16 holds). Then for all q with $|q| \leq \epsilon$,

$$\begin{aligned}
\| \sum_{r+s=k} \lambda_{r,s} \xi_{r,s} \|^2 &\geq \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |\langle \xi_{r,s}, \xi_{r',s'} \rangle| \\
&\stackrel{(20)}{\geq} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - 2 \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |q|^{t|r-r'|} [k]_q! \\
&\geq \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - 2 \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |q|^{|r-r'|} [k]_q! \\
&\stackrel{(21)}{\geq} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - 4 \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |q|^{|r-r'|} \|\xi_{r,s}\| \|\xi_{r',s'}\| \\
&= \left\langle (1 + 4E_{|q|}) \begin{pmatrix} |\lambda_{k,0}| \|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}| \|\xi_{0,k}\| \end{pmatrix}, \begin{pmatrix} |\lambda_{k,0}| \|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}| \|\xi_{0,k}\| \end{pmatrix} \right\rangle,
\end{aligned}$$

where $E_{|q|}$ is the matrix defined in the previous lemma. As q approaches 0, $4\|E_{|q|}\| \rightarrow 0$ by the previous lemma, $1 + 4E_{|q|}$ will become strictly positive once q is close enough to 0. Also, notice that the strict positivity of $1 + 4E_{|q|}$ depends on neither i nor k . This shows the existence of $A_q > 0$ satisfying the first half of (22).

Similarly,

$$\begin{aligned}
\| \sum_{r+s=k} \lambda_{r,s} \xi_{r,s} \|^2 &\leq \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |\langle \xi_{r,s}, \xi_{r',s'} \rangle| \\
&\stackrel{(20)}{\leq} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + 2 \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |q|^{t|r-r'|} [k]_q! \\
&\leq \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + 2 \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |q|^{|r-r'|} [k]_q! \\
&\stackrel{(21)}{\leq} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + 4 \sum_{r+s=r'+s'=k, r \neq r'} |\lambda_{r,s} \lambda_{r',s'}| \cdot |q|^{|r-r'|} \|\xi_{r,s}\| \|\xi_{r',s'}\| \\
&= \left\langle (1 - 4E_{|q|}) \begin{pmatrix} |\lambda_{k,0}| \|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}| \|\xi_{0,k}\| \end{pmatrix}, \begin{pmatrix} |\lambda_{k,0}| \|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}| \|\xi_{0,k}\| \end{pmatrix} \right\rangle,
\end{aligned}$$

and the existence of B_q is obvious.

The completeness is already shown in Lemma 10, therefore we are done. \square

Remark 20. By Lemma 18, $1 \pm 4E_q$ is strictly positive when $|q| < 1/9$.

Remark 21. Recall that $\xi^0 = \Omega$. If we consider $\left\{ \frac{\xi_{r,0}^0}{\|\xi_{r,0}^0\|} : r \geq 0 \right\} \cup \left\{ \frac{\xi_{r,s}^i}{\|\xi_{r,s}^i\|} : i \in I, r, s \geq 0 \right\}$, then this is a Rietz basis of the entire q -Fock space $L^2(M)$.

4 Locating the supports of elements in the relative commutant

Throughout this section we will assume that $-1 < q < 1$ is a real number with $|q|$ small enough such that the conclusions in Corollary 16 and Theorem 19 hold.

For $N \geq 0$, define the idempotent $L_N, R_N : L^2(M) \rightarrow L^2(M)$ by $L_N|_{\mathcal{F}_q(\mathbb{R}e)} = R_N|_{\mathcal{F}_q(\mathbb{R}e)} = 0$ and

$$\begin{aligned} L_N\left(\sum_{i \in I, r, s \geq 0} c_{r,s}^i \xi_{r,s}^i\right) &= \sum_{i \in I, s \geq 0, 0 \leq r \leq N} c_{r,s}^i \xi_{r,s}^i, \\ R_N\left(\sum_{i \in I, r, s \geq 0} c_{r,s}^i \xi_{r,s}^i\right) &= \sum_{i \in I, r \geq 0, 0 \leq s \leq N} c_{r,s}^i \xi_{r,s}^i. \end{aligned} \tag{23}$$

By Theorem 19, L_N and R_N are both well-defined. Moreover, with a little abuse of notation, sometimes we will also use L_N (resp. R_N) to denote the image of L_N (resp. R_N).

Let $C \subset A$ be a diffuse subalgebra and fix a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Let $z = (z_n)_n \in M^\omega \ominus A^\omega \cap C'$. Without loss of generality we assume that $\|z_n\| \leq 1$ (the operator norm is bounded above by 1) and $x_n \in M \ominus A, \forall n$. Just as in [28], we would like to show that the support of z eventually escapes both L_N and R_N . To this end, we need some preparations.

The first key step towards our goal is to show that L_N is asymptotically right- A modular.

Recall that Q_k is the orthogonal projection from $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ onto $\mathcal{H}^{\otimes k}$.

Lemma 22. *For any $k \in \mathbb{N}$, we have*

$$\lim_{n \rightarrow \omega} Q_k(z_n) = 0.$$

Proof. Suppose this is not the case, then there exists some $k \in \mathbb{N}$ with

$$\mathcal{H}^{\otimes k} \ni z_0 = \lim_{n \rightarrow \omega} Q_k(z_n) \neq 0.$$

In particular, $z_n \rightarrow z$ weakly for some non-zero $z \in M$. Clearly $z \in C' \cap M \ominus A$.

However, it is known that the generator masa A is *mixing* in M (see [2], [29]). Thus by Proposition 5.1 in [9] we must have that $C' \cap M = A$, a contradiction. \square

The next estimate will be essential in order to establish the right- A modularity of L_N .

Lemma 23. Let $x \in L^2(M) \ominus L^2(A)$ whose Fourier expansion along $\{\xi_{r,s}^i : i \in I, r, s \geq 0\}$ is of the form

$$x = \sum_{r \geq N+1, s \geq 0} \lambda_{r,s} \xi_{r,s},$$

where $\xi = \xi^i$ for some fixed $i \in I$ with $\xi \in T^t$. Then we have

$$\left\| L_N \left(a_r(e)^k x \right) \right\|_2^2 \leq \frac{4k B_q C(|q|)^3 D(q)^6}{(1-q)^k (1-q^{2t})} \sum_{r,s \geq 0} q^{2(t+s+r-k-N-1)} |\lambda_{r,s}|^2 \|\xi_{r,s}\|_2^2, \quad (24)$$

for all $k, N \geq 0$.

Proof. We let $\lambda_{r,s} = 0$ for all $r \leq N$ and $s \geq 0$. By (17), we have

$$\begin{aligned} & L_N \left(a_r(e)^k \sum_{r \geq N+1, s \geq 0} \lambda_{r,s} \xi_{r,s} \right) \\ &= L_N \left(\sum_{r \geq N+1, s \geq 0} \lambda_{r,s} \sum_{i+j=k, i, j \geq 0} \frac{[r]_q!}{[r-i]_q!} \cdot \frac{[s]_q!}{[s-j]_q!} \cdot \binom{k}{i}_q q^{(t+s-j)i} \xi_{r-i, s-j} \right) \\ &= \sum_{r \leq N, s \geq 0} \xi_{r,s} \sum_{i+j=k, r+i \geq N+1, j \geq 0} \lambda_{r+i, s+j} \frac{[r+i]_q!}{[r]_q!} \cdot \frac{[s+j]_q!}{[s]_q!} \cdot \binom{k}{i}_q q^{(t+s)i}. \end{aligned}$$

Note that for all $-1 < q < 1$,

- $\frac{[r+i]_q!}{[r]_q!} \cdot \frac{[s+j]_q!}{[s]_q!} \leq \frac{D(q)^2}{(1-q)^k}, \quad \forall i+j=k;$
- $\binom{k}{i}_q \leq D(q) C(|q|).$

Therefore, for each $r \leq N$,

$$\begin{aligned} & \left| \sum_{i+j=k, r+i \geq N+1, j \geq 0} \lambda_{r+i, s+j} \frac{[r+i]_q!}{[r]_q!} \cdot \frac{[s+j]_q!}{[s]_q!} \cdot \binom{k}{i}_q q^{(t+s)i} \right|^2 \\ & \leq \left| \sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}| \|\xi_{r+i, s+j}\|_2 \cdot \frac{1}{\|\xi_{r+i, s+j}\|_2} \frac{C(|q|) D(q)^3}{(1-q)^k} |q|^{(t+s)i} \right|^2 \\ & \leq \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}|^2 \|\xi_{r+i, s+j}\|_2^2 \right) \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} \frac{1}{\|\xi_{r+i, s+j}\|_2^2} \frac{C(|q|)^2 D(q)^6}{(1-q)^{2k}} q^{2(t+s)i} \right) \end{aligned}$$

$$\begin{aligned}
& \stackrel{(21)}{\leq} \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}|^2 \|\xi_{r+i, s+j}\|_2^2 \right) \frac{2}{[r+s+k]_q!} \cdot \frac{C(|q|)^2 D(q)^6}{(1-q)^{2k}} \sum_{i+j=k, r+i \geq N+1, j \geq 0} q^{2(t+s)i} \\
& \leq \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}|^2 \|\xi_{r+i, s+j}\|_2^2 \right) \frac{2}{[r+s+k]_q!} \cdot \frac{C(|q|)^2 D(q)^6}{(1-q)^{2k}} \cdot \frac{q^{2(t+s)(N+1-r)}}{1-q^{2(t+s)}} \\
& \leq \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}|^2 \|\xi_{r+i, s+j}\|_2^2 \right) \frac{2C(|q|)^2 D(q)^6}{(1-q)^{2k}(1-q^2)} \cdot \frac{q^{2(t+s+r-N-1)}}{[r+s+k]_q!},
\end{aligned}$$

where in the last inequality we used the fact that $ab \geq a - b$ for all $a, b \geq 1$.

Finally, we have

$$\begin{aligned}
& \left\| L_N \left(a_r(e)^k x \right) \right\|_2^2 \\
& = \left\| \sum_{r \leq N, s \geq 0} \xi_{r,s} \sum_{i+j=k, r+i \geq N+1, j \geq 0} \lambda_{r+i, s+j} \frac{[r+i]_q!}{[r]_q!} \cdot \frac{[s+j]_q!}{[s]_q!} \cdot \binom{k}{i}_q q^{(t+s)i} \right\|_2^2 \\
& \leq B_q \sum_{r \leq N, s \geq 0} \|\xi_{r,s}\|_2^2 \left| \sum_{i+j=k, r+i \geq N+1, j \geq 0} \lambda_{r+i, s+j} \frac{[r+i]_q!}{[r]_q!} \cdot \frac{[s+j]_q!}{[s]_q!} \cdot \binom{k}{i}_q q^{(t+s)i} \right|^2 \\
& \leq B_q \sum_{r \leq N, s \geq 0} \|\xi_{r,s}\|_2^2 \cdot \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}|^2 \|\xi_{r+i, s+j}\|_2^2 \right) \frac{2C(|q|)^2 D(q)^6}{(1-q)^{2k}(1-q^2)} \cdot \frac{q^{2(t+s+r-N-1)}}{[r+s+k]_q!} \\
& \stackrel{(21)}{\leq} 2B_q \sum_{r \leq N, s \geq 0} [r+s]_q! \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}|^2 \|\xi_{r+i, s+j}\|_2^2 \right) \frac{2C(|q|)^2 D(q)^6}{(1-q)^{2k}(1-q^2)} \cdot \frac{q^{2(t+s+r-N-1)}}{[r+s+k]_q!} \\
& \leq 4B_q \sum_{r \leq N, s \geq 0} \left(\sum_{i+j=k, r+i \geq N+1, j \geq 0} |\lambda_{r+i, s+j}|^2 \|\xi_{r+i, s+j}\|_2^2 \right) \frac{C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)} \cdot q^{2(t+s+r-N-1)} \\
& \leq \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)} \sum_{r, s \geq 0} q^{2(t+s+r-k-N-1)} |\lambda_{r,s}|^2 \|\xi_{r,s}\|_2^2,
\end{aligned}$$

where the second last inequality is due to the fact that

$$\frac{[r+s]_q!}{[r+s+k]_q!} \leq (1-q)^k C(|q|).$$

□

Proposition 24. For all $N, m \in \mathbb{N}$ and for any $x = (x_n) \in (L^2(M \ominus A))^\omega$ such that $\lim_{n \rightarrow \omega} Q_k(x_n) \rightarrow 0, \forall k \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \omega} \|L_N(s_r(e^{\otimes m})x_n) - s_r(e^{\otimes m})L_N(x_n)\|_2 = 0.$$

In particular, for all unitary u in the C^* -algebra $C^*(s(e))$ generated by $s(e)$, $N \in \mathbb{N}$ and $(z_n) \in M^\omega \ominus A^\omega \cap C'$,

$$\lim_{n \rightarrow \omega} \|L_N(z_n u) - L_N(z_n)u\|_2 = 0.$$

Proof. Each $s_r(e^{\otimes m})$ can be written as a finite linear combination of $c_r(e)^k a_r(e)^l$ with $k, l \geq 0$ and $k + l = m$, hence it suffices to show

$$\lim_{n \rightarrow \omega} \|L_N(c_r(e)^k a_r(e)^l x_n) - c_r(e)^k a_r(e)^l L_N(x_n)\|_2 = 0,$$

for all $k, l \geq 0, k + l = m$.

Since $c_r(e)L_N(\xi) = L_N(c_r(e)\xi)$ for all $\xi \in L^2(M \ominus A)$, it then reduces to prove

$$\lim_{n \rightarrow \omega} \|L_N(a_r(e)^k x_n) - a_r(e)^k L_N(x_n)\|_2 = 0,$$

for all $0 \leq k \leq m$.

Suppose that $x_n = \sum_{i \in I, r, s \geq 0} \lambda_{r,s}^{i,n} \xi_{r,s}^i$ is the Fourier decomposition along the Riesz basis $\{\xi_{r,s}^i\}$, observe that

$$\begin{aligned} L_N(a_r(e)^k x_n) - a_r(e)^k L_N(x_n) &= L_N(a_r(e)^k x_n) - L_N(a_r(e)^k L_N(x_n)) \\ &= L_N(a_r(e)^k (1 - L_N)(x_n)) \\ &= L_N\left(a_r(e)^k \sum_{i \in I, r \geq N+1, s \geq 0} \lambda_{r,s}^{i,n} \xi_{r,s}^i\right) \\ &= \sum_{i \in I} L_N\left(a_r(e)^k \sum_{r \geq N+1, s \geq 0} \lambda_{r,s}^{i,n} \xi_{r,s}^i\right). \end{aligned}$$

By Lemma 23, we have

$$\begin{aligned} \|L_N(a_r(e)^k x_n) - a_r(e)^k L_N(x_n)\|_2^2 &= \sum_{i \in I} \left\| L_N\left(a_r(e)^k \sum_{r \geq N+1, s \geq 0} \lambda_{r,s}^{i,n} \xi_{r,s}^i\right) \right\|_2^2 \\ &\leq \sum_{i \in I} \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k (1-q^2)} \sum_{r,s \geq 0} q^{2(|\xi^i|+s+r-k-N-1)} |\lambda_{r,s}^{i,n}|^2 \|\xi_{r,s}^i\|_2^2. \end{aligned}$$

We may assume that for each n , there exists a natural number t_n such that: (1) $t_n \rightarrow \infty$ as $n \rightarrow \omega$ and (2) $\lambda_{r,s}^{i,n} = 0$ for any i, r, s with $|\xi_i| + r + s \leq t_n + N + 1$. Thus

$$\begin{aligned} \|L_N(a_r(e)^k x_n) - a_r(e)^k L_N(x_n)\|_2^2 &\leq \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)} \cdot q^{2(t_n-k)} \sum_{i,r,s} |\lambda_{r,s}^{i,n}|^2 \|\xi_{r,s}^i\|_2^2 \\ &\leq \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)A_q} \cdot q^{2(t_n-k)} \|x_n\|_2^2. \end{aligned}$$

As $n \rightarrow \omega$, t_n diverges to infinity. Therefore the right-hand side of the above inequality converges to 0 (uniformly on the unit ball of $L^2(M^\omega \ominus A^\omega)$).

The case for $u \in C^*(s(e))$ and $z = (z_n) \in M^\omega \ominus A^\omega \cap C'$ then is an easy consequence of Lemma 22 and the fact that $\{s(e^{\otimes n}) : n \geq 0\}$ spans norm-densely in $C^*(s(e))$. \square

Our next step is to show that for any $x \in M \ominus A$ and for all sequence of unitary elements $(u_k)_k$ in $C^*(s(e))$ which goes to 0 weakly, $(u_k L_N(x))_k$ is asymptotically orthogonal to the subspace L_N .

Proposition 25. *There exists a positive number $G(q) > 0$ such that*

$$\|L_{N_1}(s(e^{\otimes n})L_{N_2}(x))\|_2 \leq G(q) \cdot (n+1)^{3/2} \cdot |q|^{(n-N_1-N_2)} \|x\|_2 \quad (25)$$

for any $N_1, N_2 \in \mathbb{N}$ and $x \in M \ominus A$. The choice of $G(q)$ is independent of N_1, N_2, x .

Proof. By the Wick formula (5), one has

$$L_{N_1}(s(e^{\otimes n})L_{N_2}(x)) = \sum_{k=0}^n \binom{n}{k}_q L_{N_1}(c(e)^k a(e)^{n-k} L_{N_2}(x)).$$

Also, notice that in the above summation, only the terms with $0 \leq k \leq N_1$ will be able to contribute something non-zero.

Let us estimate each summand for $0 \leq k \leq N_1$. Suppose $x = \sum_i \sum_{r,s} \lambda_{r,s}^i \xi_{r,s}^i$ be the Fourier decomposition along $\{\xi_{r,s}^i\}$, then

$$\begin{aligned} &L_{N_1}(c(e)^k a(e)^{n-k} L_{N_2}(x)) \\ &= \sum_i L_{N_1} \left(c(e)^k \sum_{r \leq N_2, s \geq 0} \lambda_{r,s}^i \sum_{j_1+j_2=n-k, j_2 \leq r} \frac{[r]_q!}{[r-j_2]_q!} \frac{[s]_q!}{[s-j_1]_q!} \cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-j_2)j_1} \xi_{r-j_2, s-j_1}^i \right) \\ &= \sum_i L_{N_1} \left(\sum_{r \leq N_2, s \geq 0} \lambda_{r,s}^i \sum_{j_1+j_2=n-k, j_2 \leq r} \frac{[r]_q!}{[r-j_2]_q!} \frac{[s]_q!}{[s-j_1]_q!} \cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-j_2)j_1} \xi_{r+k-j_2, s-j_1}^i \right) \end{aligned}$$

$$= \sum_i \left(\sum_{k \leq r \leq N_1, s \geq 0} \xi_{r,s}^i \sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} \lambda_{r+j_2-k, s+j_1}^i \frac{[r+j_2-k]_q!}{[r-k]_q!} \frac{[s+j_1]_q!}{[s]_q!} \cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-k)j_1} \right).$$

Now for $k \leq r \leq N_1$,

$$\begin{aligned} & \left| \sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} \lambda_{r+j_2-k, s+j_1}^i \frac{[r+j_2-k]_q!}{[r-k]_q!} \frac{[s+j_1]_q!}{[s]_q!} \cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-k)j_1} \right|^2 \\ & \leq \sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} |\lambda_{r+j_2-k, s+j_1}^i| \cdot \frac{D(q)^2}{(1-q)^{n-k}} \cdot C(|q|)D(q) \cdot |q|^{j_1} \\ & \leq \left(\sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} |\lambda_{r+j_2-k, s+j_1}^i|^2 \|\xi_{r+j_2-k, s+j_1}\|_2^2 \right) \left(\sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} \frac{1}{\|\xi_{r+j_2-k, s+j_1}\|_2^2} \frac{C(|q|)^2 D(q)^6}{(1-q)^{2(n-k)}} q^{2j_1} \right) \\ & \leq \left(\sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} |\lambda_{r+j_2-k, s+j_1}^i|^2 \|\xi_{r+j_2-k, s+j_1}\|_2^2 \right) \frac{2}{[r+s+n-2k]_q!} \frac{C(|q|)^2 D(q)^6}{(1-q)^{2(n-k)}} \cdot \frac{q^{2(n-N_1-N_2)}}{1-q^2} \\ & \leq \left(\sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} |\lambda_{r+j_2-k, s+j_1}^i|^2 \|\xi_{r+j_2-k, s+j_1}\|_2^2 \right) \frac{2C(|q|)^2 D(q)^6}{(1-q)^{2(n-k)}(1-q^2)[r+s+n-2k]_q!} \cdot q^{2(n-N_1-N_2)}. \end{aligned}$$

Hence

$$\begin{aligned} & \left\| L_{N_1} \left(c(e)^k a(e)^{n-k} L_{N_2}(x) \right) \right\|_2^2 \\ & = \sum_i \left\| \sum_{k \leq r \leq N_1, s \geq 0} \xi_{r,s}^i \sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} \lambda_{r+j_2-k, s+j_1}^i \frac{[r+j_2-k]_q!}{[r-k]_q!} \frac{[s+j_1]_q!}{[s]_q!} \cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-k)j_1} \right\|_2^2 \\ & \leq B_q \sum_i \sum_{k \leq r \leq N_1, s \geq 0} \|\xi_{r,s}^i\|_2^2 \cdot \left| \sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} \lambda_{r+j_2-k, s+j_1}^i \frac{[r+j_2-k]_q!}{[r-k]_q!} \frac{[s+j_1]_q!}{[s]_q!} \cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-k)j_1} \right|^2 \end{aligned}$$

$$\begin{aligned}
&\leq B_q \sum_i \sum_{k \leq r \leq N_1, s \geq 0} \left(\sum_{\substack{j_1+j_2=n-k, \\ j_1 \geq n-2k+r-N_2}} |\lambda_{r+j_2-k, s+j_1}^i|^2 \|\xi_{r+j_2-k, s+j_1}\|_2^2 \right) \cdot \frac{4C(|q|)^3 D(q)^6}{(1-q)^n (1-q^2)} \cdot q^{2(n-N_1-N_2)} \\
&\leq \frac{4(n+1)B_q C(|q|)^3 D(q)^6}{(1-q)^n (1-q^2) A_q} \cdot q^{2(n-N_1-N_2)} \|x\|_2^2.
\end{aligned}$$

Finally, using the rough estimate $\binom{n}{k}_q \leq C(|q|)D(q)$ and the triangle inequality, we conclude that

$$\|L_{N_1}(s(e^{\otimes n})L_{N_2}(x))\|_2 \leq (n+1)C(|q|)D(q) \sqrt{\frac{4(n+1)B_q C(|q|)^3 D(q)^6}{(1-q)^n (1-q^2) A_q}} \cdot |q|^{(n-N_1-N_2)} \|x\|_2.$$

Set $G(q) = \frac{2B_q^{1/2} C(|q|)^{5/2} D(q)^4}{(1-q)^{n/2} (1-q^2)^{1/2} A_q^{1/2}}$, we are done. \square

Finally, we are ready to prove our main result in the section.

Theorem 26. *For all $N \in \mathbb{N}$ and $z = (z_n)_n \in M^\omega \ominus A^\omega \cap C'$, we have*

$$\lim_{n \rightarrow \omega} \|L_N(z_n)\|_2 = \lim_{n \rightarrow \omega} \|R_N(z_n)\|_2 = 0. \quad (26)$$

Proof. The proof is similar to the ones in [22][28], but for completeness we include a sketch.

Fix $N \in \mathbb{N}$. Let $(u_n)_n$ be a sequence of unitary elements in C which converges to 0 weakly. Let $(u'_n)_n$ be a sequence of unitaries in $C^*(s(e))$ such that $\|u_n - u'_n\|_2 \leq \frac{1}{2^n}$. Then as in the proof of [28, Lemma 9], we further approximate u'_n with finitely supported elements in $C^*(s(e))$:

Claim: There exists a sequence (v_k) of finitely supported elements in $C^*(s(e))$ and increasing sequences of natural numbers (n_k) and (M_k) , a positive constant $H(q)$, such that

$$\|u'_{n_k} - v_k\|_2 \leq \frac{1}{2^k}, \quad \|L_N(v_k(L_{M_{k+1}} - L_{M_k})(x)) - L_N(v_k x)\|_2 < H(q)|q|^{2k/3} \|x\|_2$$

for all $x \in M \ominus A$.

Proof of the Claim. We choose (v_k) , (n_k) and (M_k) inductively. Assume that we have already chosen them up to $\{M_k, n_{k-1}, v_{k-1}\}$. Since $u_n \rightarrow 0$ weakly, if we choose a large n_k , then it is possible to well-approximate u'_{n_k} in operator-norm with a finitely supported element v_k in $C^*(s(e))$ whose support is contained in $[M_k + N + k + 1, N_k]$ for some $N_k \in \mathbb{N}$. By Proposition 25, we have $\|L_N(v_k L_{M_k}(x))\|_2 \leq G(q)|q|^{2(k+1)/3} \|x\|_2$ for all $x \in M \ominus A$, where $G(q)$ is a positive constant which only depends on q . If we take $M_{k+1} \geq N_k + N + 1$

large enough, then we have $\|L_N(v_k(L_{M_{k+1}}(x) - x))\|_2 < |q|^k$. Thus we are done.

Let us continue with the proof. On one hand, using $z \in C' \cap M^\omega$ and the asymptotic right- A modularity of L_N as in Proposition 24, we have

$$\begin{aligned}
\lim_{n \rightarrow \omega} \sum_{k=N_1}^{N_2} \langle L_N(u_{n_k} z_n), L_N(u_{n_k} z_n) \rangle &= \lim_{n \rightarrow \omega} \sum_{k=N_1}^{N_2} \langle L_N(z_n u_{n_k}), L_N(z_n u_{n_k}) \rangle \\
&\geq \lim_{n \rightarrow \omega} \sum_{k=N_1}^{N_2} \left(\langle L_N(z_n u'_{n_k}), L_N(z_n u'_{n_k}) \rangle - \|L_N(z_n(u_{n_k} - u'_{n_k}))\|_2^2 \right) \\
&\geq \lim_{n \rightarrow \omega} \sum_{k=N_1}^{N_2} \left(\langle L_N(z_n) u'_{n_k}, L_N(z_n) u'_{n_k} \rangle - \frac{\|L_N\|^2 \|z_n\|^2}{2^{2n_k}} \right) \\
&\geq (N_2 - N_1) \lim_{n \rightarrow \omega} \left(\|L_N(z_n)\|_2^2 - \frac{\|L_N\|^2 \|z_n\|^2}{2^{2N_1}} \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{n \rightarrow \omega} \sum_{k=N_1}^{N_2} \langle L_N(u_k z_n), L_N(u_k z_n) \rangle &\approx \lim_{n \rightarrow \omega} \sum_{k=N_1}^{N_2} \langle L_N(v_k(L_{M_{k+1}} - L_{M_k})(z_n)), L_N(v_k(L_{M_{k+1}} - L_{M_k})(z_n)) \rangle \\
&\leq \lim_{n \rightarrow \omega} \|L_N\|^2 \|v_k\|^2 \sum_{k=N_1}^{N_2} \langle (L_{M_{k+1}} - L_{M_k})(z_n), (L_{M_{k+1}} - L_{M_k})(z_n) \rangle \\
&\leq \frac{4B_q}{A_q} \lim_{n \rightarrow \omega} \|L_N\|^2 \|z_n\|_2^2.
\end{aligned}$$

By combining the above two estimates and by increasing $N_2 - N_1$ and N_1 , we get the conclusion for L_N . The statement about R_N follows by symmetry. \square

5 Strong asymptotic orthogonality property

Definition 27. Let $B \subset N$ be an inclusion of finite von Neumann algebras, we say that the inclusion has the *strong asymptotic orthogonality property* (s-AOP for short), if for all $a, b \in N \ominus B$ and $x = (x_n) \in N^\omega \ominus B^\omega \cap C'$, where $C \subset B$ is any diffuse subalgebra of B , then

$$ax \perp xb.$$

Fix an orthonormal basis $\{e_j : j \in J\}$ of $\mathcal{H}_{\mathbb{R}}$ with $e = e_{j_0}$ for some $j_0 \in J$.

Recall that for any $s, t \geq 0$ and for any $\xi \in L^2(M)$, we set

$$\xi_{s,t} = c(e)^s c_r(e)^t \xi.$$

The following is a direct consequence of the definition of annihilation operators:

Lemma 28. *For all $x \in L^2(M)$, $N \in \mathbb{N}$ and $j \in J \setminus \{j_0\}$,*

$$a(e_j)x_{N,N} = q^N (a(e_j)x)_{N,N}.$$

The next estimate is the key technical result of this section.

Lemma 29. *For all $x \in L^2(M \ominus A)$, $N \in \mathbb{N}$ and $j \in J \setminus \{j_0\}$, we have*

$$\|a(e_j)x_{N,N}\|_2^2 \leq \frac{16B_q^2}{A_q^2} \cdot D(q)C(|q|) \cdot q^{2N} \|a(e_j)\|_\infty \|x_{N,N}\|_2^2. \quad (27)$$

Proof. Suppose that along the Riesz basis $\{\xi_{r,s}^i : i \in I, r, s \geq 0\} \cup \{\xi_{r,0}^0 : r \geq 0\}$, we have the Fourier expansions

$$\begin{aligned} x &= \sum_{i,r,s} \lambda_{r,s}^i \xi_{r,s}^i, \\ a(e_j)x &= \sum_{i,r,s} \mu_{r,s}^i \xi_{r,s}^i + \sum_{r \geq 0} \mu_r^0 \xi_{r,0}^0. \end{aligned}$$

First note that

- $[r+s+2N]_q! \leq \frac{D(q)}{(1-q)^{2N}} \cdot [r+s]_q!$ and
- $[r+s+2N]_q! \geq \frac{1}{(1-q)^{2N}C(|q|)} \cdot [r+s]_q!.$

By the previous lemma,

$$\begin{aligned} \|a(e_j)x_{N,N}\|_2^2 &= q^{2N} \| (a(e_j)x)_{N,N} \|_2^2 \\ &= q^{2N} \left\| \sum_{i,r,s} \mu_{r,s}^i \xi_{r+N,s+N}^i + \sum_{r \geq 0} \mu_r^0 \xi_{r+2N,0}^0 \right\|_2^2 \\ &\leq q^{2N} B_q \left(\sum_{i,r,s} |\mu_{r,s}^i|^2 \|\xi_{r+N,s+N}^i\|_2^2 + \sum_{r \geq 0} |\mu_r^0|^2 \|\xi_{r+2N,0}^0\|_2^2 \right) \\ &\leq 2q^{2N} B_q \left(\sum_{i,r,s} |\mu_{r,s}^i|^2 [r+s+2N]_q! + \sum_{r \geq 0} |\mu_r^0|^2 [r+2N]_q! \right) \\ &\leq \frac{2q^{2N} B_q D(q)}{(1-q)^{2N}} \left(\sum_{i,r,s} |\mu_{r,s}^i|^2 [r+s]_q! + \sum_{r \geq 0} |\mu_r^0|^2 [r]_q! \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{4q^{2N}B_qD(q)}{(1-q)^{2N}} \left(\sum_{i,r,s} |\mu_{r,s}^i|^2 \|\xi_{r,s}^i\|_2^2 + \sum_{r \geq 0} |\mu_r^0|^2 \|\xi_{r,0}^0\|_2^2 \right) \\
&\leq \frac{4q^{2N}B_qD(q)}{(1-q)^{2N}A_q} \|a(e_j)x\|_2^2 \\
&\leq \frac{4q^{2N}B_qD(q)}{(1-q)^{2N}A_q} \|a(e_j)\|_\infty \|x\|_2^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|x_{N,N}\|_2^2 &= \left\| \sum_{i,r,s} \lambda_{r,s}^i \xi_{r+N,s+N}^i \right\|_2^2 \\
&\geq A_q \sum_{i,r,s} |\lambda_{r,s}^i|^2 \|\xi_{r+N,s+N}^i\|_2^2 \\
&\geq \frac{A_q}{2} \sum_{i,r,s} |\lambda_{r,s}^i|^2 [r+s+2N]_q! \\
&\geq \frac{A_q}{2(1-q)^{2N}C(|q|)} \sum_{i,r,s} |\lambda_{r,s}^i|^2 [r+s]_q! \\
&\geq \frac{A_q}{4(1-q)^{2N}C(|q|)} \sum_{i,r,s} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|_2^2 \\
&\geq \frac{A_q}{4(1-q)^{2N}B_qC(|q|)} \|x\|_2^2.
\end{aligned}$$

Combine these two inequalities, we have

$$\|a(e_j)x_{N,N}\|_2^2 \leq \frac{16B_q^2}{A_q^2} \cdot D(q)C(|q|) \cdot q^{2N} \|a(e_j)\|_\infty \|x_{N,N}\|_2^2.$$

□

Theorem 30. *The inclusion $A \subset M$ has the strong asymptotic orthogonality property, whenever $|q|$ is sufficiently small.*

Proof. Suppose $C \subset A$ is a diffuse subalgebra and $z = (z_n) \in M^\omega \ominus A^\omega \cap C'$ with $\|z_n\| \leq 1$. By Theorem 26, we can assume that for any $N \in \mathbb{N}$,

$$\lim_{n \rightarrow \omega} \|L_N(z_n)\|_2 = \lim_{n \rightarrow \omega} \|R_N(z_n)\|_2 = 0.$$

A density argument reduces the problem to showing that

$$\lim_{n \rightarrow \omega} \langle s(e_{i(1)} \otimes \cdots \otimes e_{i(k_1)})z_n, s_r(e_{j(1)} \otimes \cdots \otimes e_{j(k_2)})z_n \rangle = 0,$$

for all $k_1, k_2 \geq 1$ and $i(1), \dots, i(k_1), j(1), \dots, j(k_2) \in J$ such that $\{i(l) : 1 \leq l \leq k_1\} \setminus \{j_0\} \neq \emptyset$ and $\{j(l) : 1 \leq l \leq k_2\} \setminus \{j_0\} \neq \emptyset$.

By the Wick formula (5), it suffices to show the inner product

$$\langle c(e_{i(1)}) \cdots c(e_{i(t)}) a(e_{i(t+1)}) \cdots a(e_{i(k_1)}) z_n, c_r(e_{j(1)}) \cdots c_r(e_{j(s)}) a_r(e_{j(s+1)}) \cdots a_r(e_{j(k_2)}) z_n \rangle$$

goes to 0 as $n \rightarrow \omega$ for any $0 \leq t \leq k_1$ and $0 \leq s \leq k_2$. There are two cases:

Case 1: there exists some $l_1 \geq t + 1$ with $i(l_1) \neq j_0$, then the previous lemma implies that for any $N \in \mathbb{N}$, one has

$$\|c(e_{i(1)}) \cdots c(e_{i(t)}) a(e_{i(t+1)}) \cdots a(e_{i(k_1)}) z_n\|_2 \leq q^{2N} \|z_n\|_2,$$

once n gets sufficiently large (note that here we suppressed the constant before q^{2N} by increasing n if necessary). By letting $N \rightarrow \infty$, we clearly have that the inner product goes to 0 as $n \rightarrow \omega$.

Case 2: $i(t+1) = \dots = i(k_1) = j(s+1) = \dots = j(k_2) = j_0$. Let $1 \leq l_1 \leq t$ be the smallest number such that $i(l_1) \neq j_0$. Taking adjoint, we consider

$$a(e_{i(l_1)}) \cdots a(e_{i(1)}) c_r(e_{j(1)}) \cdots c_r(e_{j(s)}) a_r(e_{j(s+1)}) \cdots a_r(e_{j(k_2)}) z_n \quad (*).$$

A direct computation shows that

$$a(f_1) c_r(f_2) = c_r(f_2) a(f_1) + \langle f_1, f_2 \rangle W,$$

where $W \in B(L^2(M))$ is defined by

$$W|_{\mathcal{H}^{\otimes n}} = q^n Id, \quad \forall n \geq 0.$$

Observe also that $a(f)W = qW a(f)$.

Applying these commutation relations to $a(e_{i(l_1)}) \cdots a(e_{i(1)}) c_r(e_{j(1)}) \cdots c_r(e_{j(s)})$, we can write (*) as a finite linear combination of terms of the following form:

$$D_1 \cdots D_{m_1} a(e_{i(l_1)})^t a(e)^{m_2} a_r(e)^{k_2-s} z_n,$$

where $m_1, m_2 \geq 0, t \in \{0, 1\}$ and $D_l \in \{c_r(e_{j(1)}), \dots, c_r(e_{j(s)}), W\}, \forall 1 \leq l \leq m_1$. If $t = 1$, then we are back to Case 1;

If $t = 0$, then one of the $D_k, 1 \leq k \leq m_1$ must be W . But we know that W will decrease the size of the vector in an exponential rate with respect to the length of its basic words, so as $n \rightarrow \omega$, the length of z_n goes to infinity, thus

$$\|D_1 \cdots D_{m_1} a(e)^{m_2} a_r(e)^{k_2-s} z_n\|_2 \rightarrow 0 \quad \text{as well.}$$

□

A consequence of the theorem is a strengthening of maximal amenability:

Theorem 31. *Let $-1 < q < 1$ be a real number with $|q|$ sufficiently small, then the inclusion $A \subset M$ of the generator masa inside the q -Gaussian von Neumann algebra, has the absorbing amenability property as introduced in [8](see also [17],[28]). That is, for any diffuse subalgebra $C \subset A$, A is the unique maximal amenable extension inside M .*

Proof. A is shown to be mixing in M by [2],[29]. Thus [16, Theorem 8.1] applies. One can alternatively use the argument in [28, Proposition 1]. \square

As another application of the Rădulescu basis, we can give a very short proof of non- Γ for the q -Gaussian algebras, whenever Theorem 19 is true:

Corollary 32 (See also Avsec [1]). *Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with $\dim \mathcal{H}_{\mathbb{R}} \geq 2$. Let $-1 < q < 1$ be any real number such that Theorem 19 holds. Then $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is a full factor.*

Proof. Let $e, f \in \mathcal{H}_{\mathbb{R}}$ be two orthogonal unit vectors and let $A = \Gamma_q(\mathbb{R}e)$ (resp. $B = \Gamma_q(\mathbb{R}f)$) be the generators subalgebra associated with e (resp. f). We construct as in the previous section the Rădulescu basis $\{\xi_{r,s}^i : i \in I, r, s \geq 0\}$ respect to A . Notice that $f^{\otimes n} \in T_n$, thus we may choose the basis such that for each $n \geq 1$, $\frac{f^{\otimes n}}{\sqrt{[n]_q!}} = \xi_{0,0}^{i_n}$ for some $i_n \in I$.

Suppose $x \in M' \cap M^\omega$ with $\tau(x) = 0$. Since $x \in A' \cap M^\omega$, by applying Theorem 26 for A and by the choice of the basis, we can assume that $E_{B^\omega}(x) = 0$. Choose a Haar unitary u of A . Note that $ux = xu$ and $u \perp B$. Therefore by the s-AOP for $B \subset M$ as shown in Theorem 30, we have that $\tau(uxu^*x^*) = \tau(xx^*) = 0$. \square

Remark 33. It is already known that for all $-1 < q < 1$ and for all separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with $\dim \mathcal{H}_{\mathbb{R}} \geq 2$, $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ does not have property Γ : Avsec [1] showed the strong-solidity for all $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ and it follows from an argument in [19] that solid factors do not have property Γ .

Appendix: Strong asymptotic orthogonality property implies singularity

Proposition 34. *Let M be a finite von Neumann algebra and $A \subset M$ an abelian diffuse subalgebra. Assume that the inclusion $A \subset M$ has the strong asymptotic orthogonality property. Then A is singular in M .*

Proof. **Claim 1:** A is maximal abelian in M .

Proof of Claim 1: Let $x \in M \ominus A \cap A'$. Then when viewed as constant sequences in M^ω , we have

$$x, x^* \in M^\omega \ominus A^\omega \cap A'.$$

Hence the definition of strong asymptotic orthogonality property implies that $\tau(x^*xx^*x) = 0$, thus $x = 0$.

Now let $w \in U(M)$ be a normalizer of A . Then the von Neumann subalgebra N generated by A and w is amenable and A is a Cartan subalgebra of N .

Claim 2: If $N \neq A$, there exists a non-zero partial isometry v in N with the following properties:

- $v^*v = vv^* \in Z(N)$, where $Z(N) = N \cap N'$ is the center of N ;
- $\exists t \in \mathbb{N}$ such that $v^t = v^*v$;
- $vAv^* = Av^*v$;
- $v \perp A$.

Proof of Claim 2: Decompose N as a direct sum

$$D = \bigoplus_{n=1}^{\infty} N_{I_n} \oplus N_{II},$$

where N_{I_n} is of type I_n and N_{II} is of type II. Let z_{I_n}, z_{II} be the corresponding central projections in N . Set $A_{I_n} := Az_{I_n}, A_{II} := Az_{II}$. Then $A_{I_n} \subset N_{I_n}$ and $A_{II} \subset N_{II}$ are Cartan subalgebras.

Case 1: if for some $n \geq 2$, $z_{I_n} \neq 0$, then $A_{I_n} \subset N_{I_n}$ is of the form

$$Z(N_{I_n}) \otimes \mathbb{C}^n \subset Z(N_{I_n}) \otimes M_n(\mathbb{C}).$$

Hence $v = 1_{Z(N_{I_n})} \otimes \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \ddots \\ & & & 1 \\ 1 & & & 0 \end{pmatrix}$ does the job.

Case 2: if $z_{II} \neq 0$, then we can assume that $A_{II} \subset N_{II}$ is of the form

$$Z(N_{II}) \otimes A_0 \otimes \mathbb{C}^2 \subset Z(N_{II}) \otimes R \otimes M_2(\mathbb{C}),$$

where $A_0 \subset R$ is a Cartan subalgebra in the hyperfinite II_1 factor. Then $v =: 1 \otimes 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ works.

Set $u = v + (1 - vv^*)$, then it readily checks that u is a normalizer of A in N and $u^t = 1$. Thus u defines a $\mathbb{Z}/t\mathbb{Z}$ -action on A . Let $C := A^{Ad(u)}$ be the fixed-point subalgebra.

Then C is diffuse and $v, v^* \in M \ominus A \cap C'$. Thus by the assumption on strong asymptotic orthogonality property of $A \subset M$, we have

$$\tau(v^*vv^*v) = 0,$$

thus $v = 0$, a contradiction. Therefore, A is singular in M . □

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